Pure Multiplication Module Over the Dedekind Domain

Luthfia Nurhalimah[1, a)] Elvira Kusniyanti[1, b)], Irawati[1, c)]
[1]*Bandung Institute of Technology*

[a)] *email: luthfianurhalimah03@gmail.com*
[b)]*email: elvira@math.itb.ac.id*
[c)]*email: irawati@math.itb.ac.id*

Abstract

Various studies have explored the fascinating characteristics of modules over discrete valuation domain. One notable finding is that the multiplication module is regarded as indecomposable within a discrete valuation domain. Based on this distinctive property, a categorization of weak and pure multiplication modules over discrete valuation domain is established. A notable property of a discrete valuation ring is its role as the localization of a Dedekind domain. With this connection, there has been a classification of weak multiplication modules over the Dedekind domain. In this article, we examine the characteristics of the discrete valuation domain and the properties of pure multiplication modules over the discrete valuation domain, which collectively contribute to the properties of pure multiplication modules over the Dedekind domain.

Keywords: the Dedekind Domain, Discrete Valuation Domain, Invertible Ideal

Introduction

The ideal numbers in Ernst Kummer's study became the concepts of ideal and prime ideal in ring theory [1]. A Dedekind domain is characterized as an integral domain where any non-zero proper ideal is uniquely the result of prime ideals multiplication. Furthermore, this statement is equivalent to stating that every ideal can be inverted. In the 20th century, researchers realized that a Noetherian ring R qualifies as a Dedekind domain if and only if, for every maximal ideal P of R, the localization RP is a discrete valuation domain. In other words, a Dedekind domain is a generalization of a discrete valuation domain.

The discrete valuation domain is commonly defined by two equivalent methods, the first one being through the concept of discrete valuation, or its association with local ring [2]. This paper will utilize the properties of the discrete valuation domain related to the local domain, which subsequently highlights the significant role of the maximal ideal. The localization of a Dedekind domain is a discrete valuation domain. It has been studied that multiplication module over a Dedekind domain is indecomposable [3]. More specific, the categorization of weak and pure multiplication modules over discrete valuation domain has been accomplished. Subsequently, it has been observed that weak multiplication modules over a ring R share many similar properties as weak multiplication modules over the R localization [4]. In this paper, we search for similarity properties between pure multiplication modules over a ring R and pure multiplication modules over the R localization that may lead us to classify pure multiplication modules over the Dedekind domain related to pure multiplication modules over the discrete valuation domain classification. In this paper, the result will be provided in propositions. We begin this paper with an introduction to the discrete valuation domain.

Definition 1. An integral domain ***R*** is a discrete valuation domain if,

1. Integral domain R is a local ring.
2. Jacobson radical of R is $J(R) = pR = Rp$, with p is a non-nilpotent element of R.
3. $\cap_{n\geq 1} \left(J(R)\right)^n = 0$ for every n

[5]

A simple example of a discrete valuation domain is the localization of the ring $\mathbb{Z}$ over the set $\{\mathbb{Z}\backslash P\}$ where P is the prime ideal of R. Notated with $\mathbb{Z}_{(\mathrm{P})} = \{\frac{a}{b} | a, b \in \mathbb{Z}, b \notin P\}$.

In this research, the chosen definition of the Dedekind domain is related to the invertibility of its ideals. The invertibility of an ideal is connected to fractional ideals. An ideal I in the integral domain R is invertible if there exists a fractional ideal I^{-1} in the field of a fraction of R so that $II^{-1} = R$. In this paper, we see there is a correlation between two invertible ideals in an integral domain R.

Proposition 1. Let A and B are invertible ideals in the integral domain R. The inverse of A is $A^{-1} = \{x \in Q(R) | xA \subseteq R\}$ and the inverse of B is $B^{-1} = \{x \in Q(R) | xB \subseteq R\}$. If $A \subseteq B$ then $B^{-1} \subseteq A^{-1}$.

Proof. We take an arbitrary element $x \in B^{-1}$, then $xB \subseteq R$. We have $A \subseteq B$ so that $xA \subseteq xB \subseteq R$. It means $x \in A^{-1}$. Because for an arbitrary $x \in B^{-1}$ then $x \in A^{-1}$ it is proved that $B^{-1} \subseteq A^{-1}$.

Introduction about the ideal and its inverse has an important role in the Dedekind domain for we use a definition of the Dedekind domain related to the invertible ideal.

Definition 2. Let an R as an integral domain. If every non-zero ideal I in R is invertible then R is a Dedekind domain. [6]

The integral domain $\mathbb{Z}$ is a Dedekind domain because, for every non zero ideal $I = n\mathbb{Z}$, there always exists the fractional ideal $I^{-1} = \frac{1}{n}\mathbb{Z}$ so that $II^{-1} = \mathbb{Z}$.

As mentioned, the localization of the Dedekind domain is a discrete valuation domain. We are going to see why this is true. Let R be a Dedekind domain and P is an arbitrary maximal ideal of R. Localization of R over P notated as $R_P = \left\{\frac{a}{s} | a, s \in R, s \notin P\right\}$. Since the ideal P is also a prime ideal then $\frac{a}{s} \in R_P$ is a unit if and only if $a \in R\backslash P$. Because every element outside of P is unit then PR_P is a unique maximal ideal of R_P. Having a unique maximal ideal is equivalent to being a local ring, so R_P is a local ring. Then the Jacobson radical is the maximal ideal PR_P. Then R_P satisfied the definition of the discrete valuation domain.

Methods

The method applied by the author in this research is a literature review, where the main reference for this paper is the results of [7] study titled "Indecomposable weak multiplication modules over Dedekind domain." From the main literature, the pure multiplication module over the discrete valuation domain is classified. This paper will examine the properties of pure multiplication modules over Dedekind domain. Given the fact that the localization of Dedekind domain results in a discrete valuation domain, the properties of pure multiplication modules over a discrete valuation domain will first be explored. Subsequently, the properties of pure multiplication modules over Dedekind domain will be investigated. In this paper, all results and properties discovered are provided in the proposition. Here are some introductions to the multiplication module over discrete valuation ring and its classification from the main reference.

Definition 3. An M module over a ring R is a multiplication module if and only if for every submodule N from M there always exists an ideal I in R so that $N = IM$. [7]

Based on the given definition, a simple example of a multiplication module is $\mathbb{Z}$ as a $\mathbb{Z}$ -module. Any submodule of $\mathbb{Z}$ is an ideal in the form of $n\mathbb{Z}$ for some $n \in \mathbb{Z}$. Thus, there exists an ideal $I = n\mathbb{Z}$, which is the submodule itself so that $I\mathbb{Z} = n\mathbb{Z}$.

Next, the definition of a pure submodule is provided, along with some simple examples of pure submodules.

Definition 4. Let M be a module over the ring R. A submodule N of a module M is called a pure submodule if, for every ideal I of R, $IN = N \cap IM$. [7]

Simple examples of pure submodules include $2\mathbb{Z}$ as a submodule of $\mathbb{Z}$ as a $\mathbb{Z}$-module. Every ideal $n\mathbb{Z}$ in $2\mathbb{Z}$ equals $2n\mathbb{Z}$, which is the same as $2\mathbb{Z} \cap n\mathbb{Z}\mathbb{Z} = 2n\mathbb{Z}$. Therefore, $2\mathbb{Z}$ is a pure submodule of $\mathbb{Z}$ as a $\mathbb{Z}$-module. Similarly, every submodule of $\mathbb{Z}$ as a $\mathbb{Z}$ -module is a pure submodule.

A module is called a simple pure module if the only pure submodule is the trivial submodule. So, a module M over the ring R is a pure multiplication module if every pure submodule N from M satisfies the multiplication module property.

The pure multiplication modules over the discrete valuation domain have already been classified by Ebrahimi Atani (2008). This classification is summarized in the following theorem.

Theorem 1. If R is a discrete valuation domain with $P = \langle p \rangle$ is a maximal ideal that is unique. Then the pure multiplication R-modules are:

1. R.
2. $R/P^n, n \geq 1$.
3. $E(R/P)$, the injective hull of R/P.
4. $Q(R)$, the field of fractions of R.

[7]

Proof. This theorem is proved by first proving that each of the modules mentioned is a pure multiplication module. It is clear that R and R/P^n is a multiplication module, consequently they are pure multiplication module. And $Q(R)$ is a pure multiplication module because the only proper submodule of it is $\{0\}$ (Lu, 1995). Using the property of the annihilator of P^n it is proved that $E(R/P)$ is a pure multiplication module [9]. Then it is proved that for any pure multiplication module, it will be isomorphic to one of the modules mentioned. The condition that is considered for the arbitrary pure multiplication module over the discrete valuation domain is the height of the element and the annihilator of an arbitrary element in the module. The annihilator of an arbitrary element in the module is considered because it is related to the property that pure multiplication module over discrete valuation domain M is torsion free R-module and $P^nM = M$ $(n \geq 1)$ [10]. Let $a \in M$ with $a \neq 0$. There are two possible heights of a, the height of a is finite $(h(a) = n)$ or the height of a is infinite $(h(a) = \infty)$. Another condition considered is the annihilator. The possibilities are if the only annihilator of a notated by $(0{:}\,a) = \{r \in R | ra = 0\}$ is 0 or if the annihilator of a is the ideal P.

With the consideration of the four cases, we got these results

1. If $h(a) = n, (0{:}\,a) = P$ then $M \cong R/P^{n+1}$.
2. If $h(a) = \infty, (0{:}\,a) = 0$ then $M \cong R$.
3. If $h(a) = n, (0{:}\,a) = P$ then $M \cong Q(R)$.
4. If $h(a) = \infty, (0{:}\,a) = 0$ then $M \cong E(R/P)$.

Detailed proof is provided in [6].

With the correlation between the Dedekind domain and the discrete valuation domain, it is interesting to see if the same form of modules is also a pure multiplication module if we change the integral domain to a Dedekind domain.

Results and Discussion

Instinctively, when examining the properties of a module over a ring R, our initial step is to investigate the ring's ideal viewed as an R-module. Building on this intuition, we are currently conducting experiments on an ideal of R within the context of a Dedekind domain. Our objective is to determine whether the ideal of R, when considered as an R-module, qualifies as a pure multiplication module.

Proposition 2. Every non zero ideal of a Dedekind domain R is a pure multiplication module over R.

Proof. Consider a pure submodule N of I. Since N is a pure submodule, for every ideal J in R, it holds that $JN = N \cap JI$. We are going to find ideal L in R such that $LI = N$. Since R is a Dedekind domain, every ideal can be inverted. In other words, a fractional ideal I^{-1} will always exist such that $II^{-1} = R$. Construct the set $L = NI^{-1}$. First, it needs to be ensured that this set is ideal. Since N is a submodule of I, N is also an ideal in R, and thus, N can be inverted. In other words, there exists N^{-1} such that $N^{-1}N = R$. According to proposition 1 because $N \subseteq I$, then $I^{-1} \subseteq N^{-1}$. Therefore, $NI^{-1} \subseteq NN^{-1} \subseteq R$. It is clear that $L = NI^{-1}$ is an ideal. Thus, $LI = NI^{-1}I = NR = N$. This proves that there exists an ideal L for any pure submodule N such that $N = LI$. Therefore, I as an R-module is a pure multiplication module.

Next, observe that I, as an R-module, is isomorphic to one of the forms from the classification theorem earlier, namely R as an R-module. Since R is a Dedekind domain, $1 \in R$ can be written as $1 = ba$ with $b \in I^{-1}$ and $a \in I$. We construct isomorphism $\Phi: I \to R$ defined as follows, $\Phi(x) = bx$, where $0 \neq b \in I^{-1}$. It will be proven that the mapping Φ is an isomorphism. First, check whether Φ is a homomorphism. Take any two elements $x_1, x_2 \in I$ and any arbitrary element $r \in R$.

$$\Phi(x_1 + x_2) = b(x_1 + x_2) = bx_1 + bx_2 = \Phi(x_1) + \Phi(x_2)$$
$$r\Phi(x_1) = rbx_1 = brx_1 = \Phi(rx_1)$$

Therefore, Φ is a homomorphism. Next, prove that Φ is an injective mapping. Take two elements $bx_1, bx_2 \in R$ with $bx_1 = bx_2$. Because of this, $bx_1 - bx_2 = b(x_1 - x_2) = 0$, and since $b \neq 0$, it follows that $x_1 - x_2 = 0$, so $x_1 = x_2$. This proves that Φ is an injective mapping. Finally, it will be proven that Φ is a surjective mapping. Take any arbitrary element $r \in R$, and observe that $r = 1r$. Because I is invertible then $b \in I^{-1}$, there exists $a \in I$ such that $ba = 1$. Thus, $r = 1r = bar = \Phi(ar)$. Therefore, Φ is a surjective mapping. It is thereby proven that Φ is an isomorphism.

With this result, it is suggested that the classification of pure multiplication modules over the Dedekind domain bears a resemblance to the classification of pure multiplication modules over discrete valuation rings. To substantiate this suggestion, it will first be proven that any form of the classification of pure multiplication modules over discrete valuation rings is also a pure multiplication module when the module is over a Dedekind domain.

Proposition 3. Let R be a Dedekind domain, and let P be any maximal ideal of R. Then the following four forms are pure multiplication modules:

1. R
2. $(R/P^n), (n \geq 1)$
3. $E(R/P)$, the injective hull of (R/P)
4. $Q(R)$, the field of fractions of R

Proof.

1. Let R be a Dedekind domain. R, as an R-module is a pure multiplication module because every submodule of R is an ideal. Therefore, for any submodule S of R, it implies $S = SR$. Since R is a multiplication module over itself, it is a pure multiplication module.

2. Module (R/P^n) over R is a pure multiplication module. This can be verified by taking any pure submodule S of R/P^n. Let $S = \{0\}$, then there exists $I = P^n$ such that $(P^nR/P^n = \{0\}$. If $S \neq \{0\}$, we can construct the ideal $I = \{r \in R \mid r + P^n \in S\}$, so that $S = IR/P^n$. The construction of I ensures that $P^n R/P^n = \{0\}$. Since there always exists s an ideal I for any pure submodule S such that $IR/P^n = S$ So, R/P^n is proven to be a pure multiplication module.

3. Given R as a Dedekind domain, there is no certainty that the only potential pure submodule of $E(R/P)$ is $\{0\}$. Thus, two cases will be examined by considering any pure submodule S of $E(R/P)$. Let $S = \{0\}$ then S satisfies the multiplication property. If $S \neq \{0\}$ an ideal $I = \{r \in R \mid rE(R/P) \subseteq S\}$ can be constructed. It is evident that $IE(R/P) \subseteq S$. To check if $S \subseteq IE(R/P)$, in a Dedekind domain, every injective module is divisible. This means that for any element $s \in S$, there exists $y \in E(R/P)$ such that $s = ry$ for some $r \in R$. Since $ry \in S$, it implies $r \in I$. or $ry = s \in IE(R/P)$. Thus, $S = IE(R/P)$. Therefore, for any pure multiplication module S, there exists an ideal I such that $S = IE(R/P)$, establishing that $E(R/P)$ is a pure multiplication module.

4. The module $Q(R)$ over R is a pure multiplication module because there is no other pure submodule besides $\{0\}$. Assume there is another non-zero pure submodule $S \neq \{0\}$. Then $(S : Q(R)) = \{r \in R \mid rQ(R) \subseteq S\} = \{0\}$ because $rQ(R) = Q(R)$ for every non-zero element $r \in R$. Therefore, the ideal that satisfies $IS = S \cap IQ(R)$ can only be $\{0\}$, contradicting the definition of a pure submodule. Hence, the only pure submodule of $Q(R)$ is $\{0\}$.

For indecomposable weak multiplication modules over the Dedekind domain, it is proved that the weak multiplication modules over the Dedekind domain localization is also indecomposable [4]. In pure multiplication modules over discrete valuation rings, every non-zero module is indecomposable [7]. It turns out that this property is also satisfied in pure multiplication modules over the Dedekind domain.

Proposition 4. Every pure multiplication module over a Dedekind domain R that is non zero is also an indecomposable module.

Proof. Suppose M is a pure multiplication R-module with $M = N \oplus P$ where $N \neq 0$ and $P \neq 0$. Since the direct sum of M is a pure submodule, there exists an ideal I such that $IM = N$.

Because R is a Dedekind domain, there exists a fractional ideal I^{-1} such that $II^{-1} = R$. Multiplying the equation $M = N \oplus P$ by I^{-1} yields:

$$I^{-1}M = I^{-1}N \oplus I^{-1}P$$
$$I^{-1}M = I^{-1}IM \oplus I^{-1}P$$
$$I^{-1}M = M \oplus I^{-1}P$$

With M isomorphic to $I^{-1}M$, it can be concluded that $I^{-1}P = 0$. Since I^{-1} is invertible it follows that $P = 0$. This is contradictory to $P \neq 0$, so it must be that M is indecomposable.

Conclusion

Based on the results and discussion above, two properties of pure multiplication module over the Dedekind Domain are confirmed. We confirmed that the pure multiplication module over the Dedekind domain is indecomposable and the classification form of the pure multiplication module over the discrete valuation domain is also a pure multiplication over the Dedekind domain.

These findings make it interesting to examine whether the classification of pure multiplication over the Dedekind domain shares similarities with its classification over discrete valuation domain. Furthermore, by applying various approaches and exploring alternative definitions of the Dedekind domain, one can uncover distinct properties of pure multiplication modules over the Dedekind domain.

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